

An integrable (2+1)-dimensional Camassa-Holm hierarchy with peakon solutions

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Abstract

In this letter, we propose a (2+1)-dimensional generalized Camassa-Holm (2dgCH) hierarchy with both quadratic and cubic nonlinearity. The Lax representation and peakon solutions for the 2dgCH system are derived.

Keywords: Camassa-Holm (CH) equation, 2dgCH, Peakon, Lax representation.

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1 Introduction

In recent years, the Camassa-Holm (CH) equation [1]

$$m_t - \alpha u_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \quad (1)$$

has attracted much attention in the theory of integrable systems and solitons. Since the work of Camassa and Holm [1], various studies on this equation have remarkably been developed [5]-[12]. The most remarkable feature of the CH equation (1) is that it admits peaked soliton (peakon) solutions in the case of $\alpha = 0$ [1, 2]. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [8]-[12].

In addition to the CH equation being an integrable model with peakon solutions, other integrable peakon models have been found, including the Degasperis-Procesi (DP) equation [13] whose Lax pair, bi-Hamiltonian formulation and peakon solutions were discovered in [14, 15], the cubic nonlinear peakon equations [16, 17, 18], and a generalized CH equation (gCH) with both quadratic and cubic nonlinearity [19]

$$m_t = \frac{1}{2}k_1 [m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_x u), \quad m = u - u_{xx}, \quad (2)$$

where k_1 and k_2 are two arbitrary constants. Through some appropriate rescaling, equation (2) could be transformed to the one in the papers of Fokas and Fuchssteiner [3, 4], where it was derived from the motion of a two-dimensional, inviscid, incompressible fluid over a flat bottom. In [19], the Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interactional and smooth soliton solutions of equation (2) are presented.

It is an interesting task to study the (2+1)-dimensional generalizations of the peakon equations. For example, in [20, 21] the authors provided a (2+1)-dimensional extension of the CH hierarchy, and they further studied the hodograph transformations and peakon solutions for their (2+1)-dimensional CH equation. In the present letter, we generalize the gCH equation (2) to the whole integrable hierarchies in (1+1) and (2+1)-dimensions. We show that the gCH hierarchies admit Lax representations and construct a relation between the gCH hierarchies in (1+1) and (2+1)-dimensions. Moreover, we derive the single-peakon solution and the multi-peakon dynamic system for the (2+1)-dimensional gCH equation.

The letter is organized as follows. In section 2, we review the CH hierarchies in (1+1) and (2+1)-dimensions. In section 3, we present the gCH hierarchies in (1+1) and (2+1)-dimensions. In particular, we give their Lax representations. In section 4, we derive the peakon solutions to the (2 + 1)-dimensional gCH equation. Some conclusions and discussions are drawn in section 5.

2 Overviews

In this section, we review the (1+1) and (2+1)-dimensional CH hierarchies presented in [7, 20, 21]. The new results we find are a relation between the CH hierarchies in (1+1) and (2+1)-dimensions and an isospectral Lax representations for the CH hierarchies.

2.1 The CH hierarchies in (1+1) and (2+1)-dimensions

Let us consider the Lenard operators pair [1]

$$J = \partial_x m + m \partial_x, \quad K = \frac{1}{2}(\partial_x^3 - \partial_x). \quad (3)$$

The Lenard gradients b_{-k} are defined recursively by

$$Kb_{-k} = Jb_{-k+1}, \quad Kb_0 = 0, \quad k \in \mathbb{Z}^+. \quad (4)$$

Taking an initial value $b_0 = -\frac{1}{2}$, one may generate the negative CH hierarchy [7]

$$\begin{cases} m_{t_{-n}} = Jb_{-n}, \\ Kb_{-j} = Jb_{-j+1}, \end{cases} \quad 1 \leq j \leq n. \quad (5)$$

For $n = 1$, (5) becomes

$$\begin{cases} m_{t_{-1}} = (mb_{-1})_x + mb_{-1,x}, \\ \frac{1}{2}(b_{-1,xxx} - b_{-1,x}) = -\frac{1}{2}m_x, \end{cases} \quad (6)$$

which is nothing but the CH equation (1) with $\alpha = 0$ [1]. For $n = 2$, we arrive at

$$\begin{cases} m_{t_{-2}} = (mb_{-2})_x + mb_{-2,x}, \\ \frac{1}{2}(b_{-2,xxx} - b_{-2,x}) = (mb_{-1})_x + mb_{-1,x}, \\ \frac{1}{2}(b_{-1,xxx} - b_{-1,x}) = -\frac{1}{2}m_x. \end{cases} \quad (7)$$

In what follows, we call equation (7) the 2-nd CH equation. For the general n , we refer to (5) as the n -th CH equation.

In [20, 21], the authors proposed a (2+1)-dimensional CH equation

$$\begin{cases} m_t = (mb_{-2})_x + mb_{-2,x}, \\ \frac{1}{2}(b_{-2,xxx} - b_{-2,x}) = m_y. \end{cases} \quad (8)$$

In general, a (2+1)-dimensional generalization of the CH hierarchy could be written as [20, 21]

$$\begin{cases} m_{t_{-n}} = Jb_{-n}, \\ Kb_{-j} = Jb_{-j+1}, \quad 3 \leq j \leq n. \\ Kb_{-2} = m_y, \end{cases} \quad (9)$$

In [20, 21], the authors also studied the hodograph transformations and the peakon solutions of the (2+1)-dimensional CH equation.

2.2 Lax representation

Let

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix}, \quad V^{(-n)} = -\frac{1}{2}U + \sum_{i+j=n, 0 \leq i \leq n-1, 1 \leq j \leq n} \lambda^{-i} \tilde{V}^{(-j)}, \quad (10)$$

where

$$\tilde{V}^{(-j)} = \begin{pmatrix} -\frac{1}{2}b_{-j,x} & b_{-j} + \frac{1}{2} - \frac{1}{2\lambda} \\ m(b_{-j} + \frac{1}{2})\lambda - \frac{1}{2}b_{-j,xx} + \frac{1}{4}(b_{-j} + \frac{1}{2}) - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-j,x} \end{pmatrix}, \quad (11)$$

λ is the eigenparameter and b_j is defined through equation (4).

By a direct calculation, we obtain the following result.

Proposition 1 *The n -th CH equation (5) admits the Lax representation*

$$U_{t-n} - V_x^{(-n)} + [U, V^{(-n)}] = 0, \quad (12)$$

where the Lax pair U and $V^{(-n)}$ given by (10).

As $n = 1$, we recover the Lax pair of the well-known CH equation (1) with $\alpha = 0$ [1]

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix}, \quad V^{(-1)} = \begin{pmatrix} -\frac{1}{2}b_{-1,x} & b_{-1} - \frac{1}{2\lambda} \\ mb_{-1}\lambda - \frac{1}{2}b_{-1,xx} + \frac{1}{4}b_{-1} - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-1,x} \end{pmatrix}. \quad (13)$$

As $n = 2$, we obtain the Lax pair of the 2-nd CH equation (7)

$$\begin{aligned} U &= \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix}, \\ V^{(-2)} &= \begin{pmatrix} -\frac{1}{2}b_{-2,x} & b_{-2} - \frac{1}{2\lambda} \\ mb_{-2}\lambda - \frac{1}{2}b_{-2,xx} + \frac{1}{4}b_{-2} - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-2,x} \end{pmatrix} \\ &\quad + \frac{1}{\lambda} \begin{pmatrix} -\frac{1}{2}b_{-1,x} & b_{-1} + \frac{1}{2} - \frac{1}{2\lambda} \\ m(b_{-1} + \frac{1}{2})\lambda - \frac{1}{2}b_{-1,xx} + \frac{1}{4}(b_{-1} + \frac{1}{2}) - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-1,x} \end{pmatrix}. \end{aligned} \quad (14)$$

It has been known that there exist some relations between integrable models in (1+1)-dimensions and ones in (2+1)-dimensions. For example, assembling of the first two 1+1 dimensional non-trivial members in the AKNS hierarchy: the coupled nonlinear Schrödinger equation and the coupled mKdV equation, yields the well-known (2+1)-dimensional KP equation [22]-[25]. The compatible solution of the first two members in the KdV hierarchy produces a special solution of the (2+1)-dimensional Sawada-Kotera equation [26]-[28]. Here in our paper, we have some similar results listed as follows.

Proposition 2 *Let $t_{-1} = y$, $t_{-2} = t$. Let $m(x, y, t)$ be a compatible solution of the CH equation (6) and the 2-nd CH equation (7). Then $m(x, y, t)$ provides a special solution to (2+1)-dimensional CH equation (8). In general, if $m(x, t_{-1}, t_{-n})$ is a compatible solution of the CH equation (6) and the n -th CH equation (5), then the (2+1)-dimensional CH hierarchy (9) has a special solution $m(x, t_{-1}, t_{-n})$.*

The above proposition immediately yields the following corollary.

Corollary 1 *The (2+1)-dimensional CH equation (8) possesses a Lax triad U , $V^{(-1)}$, $V^{(-2)}$. In general, the (2+1)-dimensional CH hierarchy (9) possesses a Lax triad U , $V^{(-1)}$, $V^{(-n)}$.*

Remark 1. Based on proposition 2, we may construct the algebraic-geometric solution of the (2+1)-dimensional CH hierarchy with the method developed in [25, 26, 7]. We will consider this topic in another publication.

3 The gCH hierarchies in (1+1) and (2+1)-dimensions

Let us first introduce a pair of Lenard operators [19]

$$J = k_1 \partial_x m \partial_x^{-1} m \partial_x + \frac{1}{2} k_2 (\partial_x m + m \partial_x), \quad K = \partial_x - \partial_x^3, \quad (15)$$

and define the Lenard gradients b_{-k} recursively by

$$K b_{-k} = J b_{-k+1}, \quad K b_0 = 0, \quad k \in \mathbb{Z}^+. \quad (16)$$

We define a gCH hierarchy in (1+1)-dimension as follows

$$\begin{cases} m_{t_{-n}} = J b_{-n}, \\ K b_{-j} = J b_{-j+1}, \quad 2 \leq j \leq n. \\ K b_{-1} = m_x, \end{cases} \quad (17)$$

The first member in (17) reads as

$$\begin{cases} m_{t_{-1}} = \frac{1}{2} k_1 [m(b_{-1}^2 - b_{-1,x}^2)]_x + \frac{1}{2} k_2 (2m b_{-1,x} + m_x b_{-1}), \\ m = b_{-1} - b_{-1,xx}, \end{cases} \quad (18)$$

which is nothing but the gCH equation (2). For $n = 2$, equation (17) is cast into the 2-nd gCH equation in the gCH hierarchy (17)

$$\begin{cases} m_{t_{-2}} = k_1 [m \partial_x^{-1} m b_{-2,x}]_x + \frac{1}{2} k_2 (2m b_{-2,x} + m_x b_{-2}), \\ b_{-2,x} - b_{-2,xxx} = \frac{1}{2} k_1 [m(b_{-1}^2 - b_{-1,x}^2)]_x + \frac{1}{2} k_2 (2m b_{-1,x} + m_x b_{-1}), \\ m = b_{-1} - b_{-1,xx}. \end{cases} \quad (19)$$

For the general case $n \geq 2$, we refer to (17) as the n -th gCH equation.

Similar to the (2+1)-dimensional generalization of the CH equation, we extend the (1+1)-dimensional gCH equation (2) to the (2+1)-dimensional system as follows:

$$\begin{cases} m_t = k_1 [m \partial_x^{-1} m b_{-2,x}]_x + \frac{1}{2} k_2 (2m b_{-2,x} + m_x b_{-2}), \\ m_y = b_{-2,x} - b_{-2,xx}. \end{cases} \quad (20)$$

Furthermore, we may define the (2+1)-dimensional gCH hierarchy in the following form:

$$\begin{cases} m_{t-n} = J b_{-n}, \\ K b_{-j} = J b_{-j+1}, \quad 3 \leq j \leq n. \\ m_y = K b_{-2}, \end{cases} \quad (21)$$

In particular, as $k_1 = 0$ and $k_2 = 2$, our (2+1)-dimensional gCH hierarchy (21) is reduced to the (2+1)-dimensional CH hierarchy (9).

Let us now show that the gCH hierarchies admit Lax representations. Let

$$U = \frac{1}{2} \begin{pmatrix} -1 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 1 \end{pmatrix}, \quad V^{(-n)} = U + \sum_{0 \leq j \leq n-1} \lambda^{-2j} \tilde{V}^{-(n-j)}, \quad (22)$$

where

$$\tilde{V}^{(-j)} = -\frac{1}{2} \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (23)$$

with

$$\begin{aligned} A &= \lambda^{-2} + k_1 \partial^{-1} m b_{-j,x} + \frac{1}{2} k_2 (b_{-j} - b_{-j,x}) - 1, \\ B &= -\lambda^{-1} (m - b_{-j,x} + b_{-j,xx}) + \lambda m (-k_1 \partial^{-1} m b_{-j,x} - \frac{1}{2} k_2 b_{-j} + 1), \\ C &= \lambda^{-1} [k_1 (m + b_{-j,xx} + b_{-j,x}) + k_2] - \lambda (k_1 m + k_2) (-k_1 \partial^{-1} m b_{-j,x} - \frac{1}{2} k_2 b_{-j} + 1). \end{aligned} \quad (24)$$

Direct calculations lead to the following proposition.

Proposition 3 *The gCH hierarchy (17) possesses the Lax representation*

$$U_{t-n} - V_x^{(-n)} + [U, V^{(-n)}] = 0,$$

with the Lax pair U and $V^{(-n)}$ given by (22).

In particular, the Lax pair of the gCH equation (18) are given by

$$U = \frac{1}{2} \begin{pmatrix} -1 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 1 \end{pmatrix}, \quad V^{(-1)} = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix}, \quad (25)$$

with

$$\begin{aligned} A_1 &= \lambda^{-2} + \frac{1}{2}k_1(b_{-1}^2 - b_{-1,x}^2) + \frac{1}{2}k_2(b_{-1} - b_{-1,x}), \\ B_1 &= -\lambda^{-1}(b_{-1} - b_{-1,x}) - \frac{1}{2}\lambda m [k_1(b_{-1}^2 - b_{-1,x}^2) + k_2b_{-1}], \\ C_1 &= \lambda^{-1}[k_1(b_{-1} + b_{-1,x}) + k_2] + \frac{1}{2}\lambda [k_1^2m(b_{-1}^2 - b_{-1,x}^2) + k_1k_2(mb_{-1} + b_{-1}^2 - b_{-1,x}^2) + k_2^2b_{-1}]. \end{aligned} \quad (26)$$

The Lax pair of the 2-nd gCH equation (19) are given by

$$U = \frac{1}{2} \begin{pmatrix} -1 & \lambda m \\ -k_1\lambda m - k_2\lambda & 1 \end{pmatrix}, \quad V^{(-2)} = U + \tilde{V}^{(-2)} + \lambda^{-2}\tilde{V}^{(-1)}, \quad (27)$$

where $\tilde{V}^{(-1)}$ and $\tilde{V}^{(-2)}$ are defined by (23) and (24).

One may easily check the following results.

Proposition 4 *Let $t_{-1} = y$, $t_{-2} = t$. Let $m(x, y, t)$ be a compatible solution of the gCH equation (18) and the 2-nd gCH equation (19). Then $m(x, y, t)$ provides a special solution to (2+1)-dimensional gCH equation (20). In general, if $m(x, t_{-1}, t_{-n})$ is a compatible solution of the gCH equation (18) and the n -th gCH equation (17), then the (2+1)-dimensional gCH hierarchy (21) has a special solution $m(x, t_{-1}, t_{-n})$.*

Corollary 2 *The (2+1)-dimensional gCH equation (20) possesses the Lax triad U , $V^{(-1)}$, $V^{(-2)}$ given by (25) and (27). In general, the (2+1)-dimensional gCH hierarchy (21) possesses the Lax triad U , $V^{(-1)}$, $V^{(-n)}$ given by (22).*

4 Peakon solutions to the 2dgCH equation (20)

Assume the single-peakon solution of (2+1)-dimensional gCH equation (20) is given in the form of

$$b_{-2} = p(y, t)e^{-|x - q(y, t)|}, \quad m = 2r(y, t)\delta(x - q(y, t)), \quad (28)$$

where $p(y, t)$, $q(y, t)$ and $r(y, t)$ are to be determined. Substituting (28) into (20) and integrating against the test function with support around the peak, we finally arrive at

$$\begin{cases} r_y = r_t = 0, \\ q_y = -\frac{p}{r}, \\ q_t = -\frac{1}{3}k_1rp - \frac{1}{2}k_2p, \end{cases} \quad (29)$$

which yields

$$\begin{cases} r = c, \\ q = F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t), \\ p = -cq_y, \end{cases} \quad (30)$$

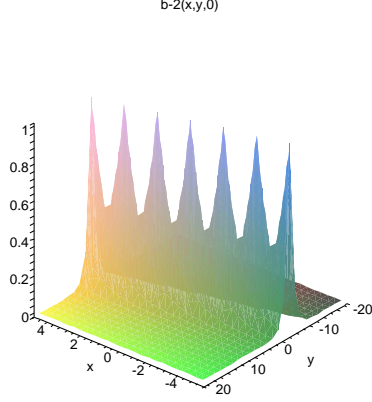


Figure 1: Single-peakon solution $b_{-2}(x, y, t)$ in (32) with $c = -1$ at $t = 0$.

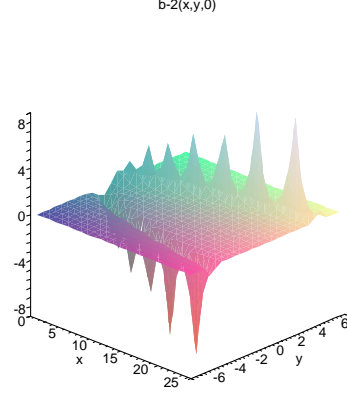


Figure 2: Single-peakon solution $b_{-2}(x, y, t)$ in (33) with $c = -1$ at $t = 0$.

where c is an arbitrary constant, F is an arbitrary smooth function. Thus, the single-peakon solution of equation (20) is given by

$$\begin{aligned} b_{-2} &= -cF_y \left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c \right)t \right) e^{-|x - F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t)|}, \\ m &= 2c\delta \left(x - F \left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c \right)t \right) \right). \end{aligned} \quad (31)$$

As $k_1 = 0$, $k_2 = 2$, we recover the single-peakon solution of the $(2+1)$ -dimensional CH equation proposed in [20].

In particular, if we take $F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t) = y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t$, then the single-peakon solution of equation (20) becomes

$$\begin{aligned} b_{-2} &= -ce^{-|x - y - (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t|}, \\ m &= 2c\delta \left(x - y - \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c \right)t \right). \end{aligned} \quad (32)$$

See Figure 1 for the graph of the single-peakon solution $b_{-2}(x, y, t)$ at $t = 0$. If we take $F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t) = (y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t)^2$, then the single-peakon solution (31) becomes

$$\begin{aligned} b_{-2} &= -2c \left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c \right)t \right) e^{-|x - (y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t)^2|}, \\ m &= 2c\delta \left(x - \left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c \right)t \right)^2 \right). \end{aligned} \quad (33)$$

See Figure 2 for the graph of $b_{-2}(x, y, t)$ in (33) at $t = 0$.

In general, let us suppose that the N -peakon has the following form

$$b_{-2} = \sum_{j=1}^N p_j(y, t) e^{-|x - q_j(y, t)|}, \quad m = 2 \sum_{j=1}^N r_j(y, t) \delta(x - q_j(y, t)). \quad (34)$$

Similar to the cases of one-peakon but with a lengthy calculation, we are able to obtain the following N -peakon dynamical system

$$\begin{aligned}
r_{j,y} &= 0, \\
r_{j,t} &= -\frac{1}{2}k_2r_j \sum_{k=1}^N p_k \operatorname{sgn}(q_j - q_k) e^{-|q_j - q_k|}, \\
p_j &= -r_j q_{j,y}, \\
q_{j,t} &= \frac{1}{6}k_1 r_j p_j - \frac{1}{2}k_2 \sum_{k=1}^N p_k e^{-|q_j - q_k|} + \frac{1}{2}k_1 \sum_{i,k=1}^N r_i p_k (\operatorname{sgn}(q_j - q_i) \operatorname{sgn}(q_j - q_k) - 1) e^{-|q_j - q_i| - |q_j - q_k|}.
\end{aligned} \tag{35}$$

5 Conclusions and discussions

In this letter, we have extended the gCH equation to the hierarchies in (1+1)-dimensions and (2+1)-dimensions. We first show the gCH hierarchies admit Lax representation. Then we show the (2+1)-dimensional gCH equation possesses single peakon solution as well as multi-peakon solutions. Other topics, such as smooth soliton solutions, cuspons, peakon stability, and algebraic-geometric solutions, remain to be developed.

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References

- [1] R. Camassa and D.D. Holm, Phys. Rev. Lett. 71 1661-1664 (1993).
- [2] R. Camassa, D.D. Holm and J.M. Hyman, Adv. Appl. Mech. 31 1-33 (1994).
- [3] A.S. Fokas, Physica D 87 145-150 (1995).

- [4] B. Fuchssteiner, *Physica D* 95 229-243 (1996).
- [5] P. J. Olver and P. Rosenau, *Phys. Rev. E* 53 1900-1906 (1996).
- [6] F. Gesztesy and H. Holden, *Rev. Mat. Iberoamericana* 19 73-142 (2003).
- [7] Z.J. Qiao, *Commun. Math. Phys.* 239 309-341 (2003).
- [8] A. Constantin and W.A. Strauss, *Comm. Pure Appl. Math.* 53 603-610 (2000).
- [9] A. Constantin and W.A. Strauss, *J. Nonlinear Sci.* 12 415-422 (2002).
- [10] R. Beals, D. Sattinger and J. Szmigielski, *Adv. Math.* 140 190-206 (1998).
- [11] R. Beals, D. Sattinger and J. Szmigielski, *Adv. Math.* 154 229-257 (2000).
- [12] R.S. Johnson, *Proc. R. Soc. Lond. A* 459 1687-1708 (2003).
- [13] A. Degasperis and M. Procesi, Asymptotic integrability, in *Symmetry and Perturbation Theory*, ed. A. Degasperis and G. Gaeta, World Scientific pp. 23-37 (1999).
- [14] A. Degasperis, D.D. Holm and A.N.W. Hone, *Theoret. Math. Phys.* 133 1463-1474 (2002).
- [15] A. Degasperis, D. D. Holm and A. N. W. Hone, Integrable and nonintegrable equations with peakons, in *Nonlinear physics: theory and experiment, II (Gallipoli, 2002)*, ed. M. J. Ablowitz, M. Boiti, F. Pempinelli and B. Prinari, World Scientific pp. 37-43 (2003).
- [16] Z.J. Qiao, *J. Math. Phys.* 47 112701 (2006), *J. Math. Phys.* 48 082701 (2007).
- [17] V. Novikov, *J. Phys. A: Math. Theor.* 42 342002 (2009).
- [18] A.N.W. Hone and J.P. Wang, *J. Phys. A: Math. Theor.* 41 372002 (2008).
- [19] Z.J. Qiao, B.Q. Xia and J.B. Li, arXiv:1205.2028v2.
- [20] P.G. Estévez and J. Prada, *Theor. Math. Phys.* 144 1132-1137 (2005).
- [21] P.G. Estévez and J. Prada, *J. Phys. A*, 38, 1287-1297 (2005).
- [22] B. Konopelchenko, J. Sidorenko, and W. Strampp, *Phys. Lett. A* 157 17-21 (1991).
- [23] Y. Cheng and Y. S. Li, *Phys. Lett. A* 157 22-26 (1991).
- [24] Y. Cheng and Y. S. Li, *J. Phys. A* 25 419-431 (1992).
- [25] C.W. Cao, Y.T. Wu and X.G. Geng, *J. Math. Phys.* 40 3948-3970 (1999).
- [26] C.W. Cao and X. Yang, *Commun. Theor. Phys.* 49 31-36 (2008).
- [27] B. Konopelchenko and V. Dubrovsky, *Phys. Lett. A* 102 15-17 (1984).
- [28] K. Sawada and J. Kotera, *Prog. Theor. Phys.* 51 1355-1367 (1974).